

Non-singlet Q-deformation of the $\mathcal{N}=(1,1)$ gauge multiplet in harmonic superspace

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Abstract

We study a non-anticommutative chiral non-singlet deformation of the $\mathcal{N}=(1,1)$ abelian gauge multiplet in Euclidean harmonic superspace with a product ansatz for the deformation matrix, $C_{(ik)}^{(\alpha\beta)} = c^{(\alpha\beta)} b_{(ik)}$. This choice allows us to obtain in closed form the gauge transformations and the unbroken $\mathcal{N}=(1,0)$ supersymmetry transformations preserving the Wess-Zumino gauge, as well as the bosonic sector of the $\mathcal{N}=(1,0)$ invariant action. This should be contrasted with the generic choice for which the analogous results are known only to a few orders in the deformation parameters. As in the case of a singlet deformation, the bosonic action can be cast in a form where it differs from the free action merely by a scalar factor. The latter is now given by $\cosh^2(2\bar{\phi}\sqrt{c^{\alpha\beta}c_{\alpha\beta}b^{ik}b_{ik}})$, with $\bar{\phi}$ being one of two scalar fields of the $\mathcal{N}=(1,1)$ vector multiplet. We compare our results with previous studies of non-singlet deformations, including the degenerate case $b^{(ik)}b_{(ik)} = 0$ which preserves the $\mathcal{N}=(1, \frac{1}{2})$ fraction of $\mathcal{N}=(1,1)$ supersymmetry.

1 Introduction

Non-anticommutative —or nilpotent— deformations of Euclidean superspaces naturally emerge in string theory while considering a low-energy limit of some superstrings in special backgrounds, e.g. a graviphoton background [1, 2]. These string theory-inspired deformations have recently boosted the interest in *non-anticommutative* Euclidean supersymmetric field theories, with properly broken $\mathcal{N} = (\frac{1}{2}, \frac{1}{2})$ or $\mathcal{N} = (1, 1)$ supersymmetries [3] - [16].¹ In the superfield approach, the nilpotent deformations are introduced via Weyl-Moyal product with a bilinear Poisson operator which is constructed either in terms of the supercharges, or in terms of the spinor covariant derivatives [17, 18, 19]. These two options lead to the Q- or D-deformation, respectively. Q-deformations break supersymmetry but preserve chirality and, in the $\mathcal{N} = (1, 1)$ case, also Grassmann harmonic analyticity. On the other hand, D-deformations are supersymmetry-preserving, but they break chirality and, generically, Grassmann harmonic analyticity. Since it is Q-deformations that are directly implied by string theory (at least for a few well elaborated cases), it seems tempting to continue studying for a while their various options and physical consequences thereof, leaving aside the issue of the precise relation of these options to specific string backgrounds. As pointed out in [10], these deformations can e.g. provide a new mechanisms of soft supersymmetry breaking in some realistic supersymmetric theories. From the mathematical point of view, they are also of interest, giving rise to a new variant of q-deformed supersymmetry [20, 21]. Their impact on the target space geometries of supersymmetric sigma models and corresponding scalar potentials was studied e.g. in [22].

In this paper we shall deal with the $\mathcal{N} = (1, 1) \rightarrow \mathcal{N} = (1, 0)$ supersymmetry breaking Q-deformations, the associated Poisson operator of which reads

$$P = -\overleftarrow{Q}_\alpha^i C_{ik}^{\alpha\beta} \overrightarrow{Q}_\beta^k. \quad (1.1)$$

The Moyal product of two superfields is then defined by

$$A \star B = A e^P B. \quad (1.2)$$

The deformation parameters $C_{ij}^{\alpha\beta}$ form a constant tensor which is symmetric under the simultaneous permutation of the Latin and Greek indices, $C_{ij}^{\alpha\beta} = C_{ji}^{\beta\alpha}$. Generically, it breaks the full automorphism symmetry $Spin(4) \times O(1,1) \times SU(2) \equiv SU(2)_L \times SU(2)_R \times O(1,1) \times SU(2)$ of the $\mathcal{N} = (1, 1)$ superalgebra ($O(1,1)$ and $SU(2)$ are the R-symmetry groups) down to $SU(2)_R$. The operator (1.1) can be split (see for example [19, 5]) as follows

$$P = -I \overleftarrow{Q}_\alpha^i \varepsilon^{\alpha\beta} \varepsilon_{ik} \overrightarrow{Q}_\beta^k - \overleftarrow{Q}_\alpha^i \hat{C}_{ik}^{\alpha\beta} \overrightarrow{Q}_\beta^k. \quad (1.3)$$

¹A more complete list of references can be found e.g. in [12].

The first term is $Spin(4) \times SU(2)$ -preserving while the second term involves a $SU(2)_L \times SU(2)$ constant tensor which is symmetric under the independent permutations of Latin and Greek indices, $\hat{C}_{ij}^{\alpha\beta} = \hat{C}_{ij}^{\beta\alpha} = \hat{C}_{ji}^{\alpha\beta}$. For the generic choice, it fully breaks “Lorentz” symmetry $SU(2)_L$ and R-symmetry $SU(2)$. Q-deformations induced by the first term only are called *singlet* or *QS-deformations*, whereas those associated with the second term can naturally be named *non-singlet* or *QNS-deformations*. A key feature of Q-deformations and D-deformations is the nilpotent nature of the Poisson operator (1.1) which advantageously makes the Moyal product polynomial. In the case under consideration $P^5 = 0$ and, as a result,

$$A \star B = AB + APB + \frac{1}{2}AP^2B + \frac{1}{6}AP^3B + \frac{1}{24}AP^4B. \quad (1.4)$$

A detailed treatment of this Poisson operator in the harmonic superspace approach was given in [5, 9]. The Moyal product associated with it breaks $\mathcal{N} = (1, 1)$ supersymmetry down to $\mathcal{N} = (1, 0)$ (P does not commute with $\bar{Q}_{\dot{\alpha}i}$) but preserves both the chirality and Grassmann harmonic analyticity of the involved superfields, as well as the harmonic conditions $D^{\pm\pm}A = 0$, in virtue of the properties

$$[D_{\alpha}^{\pm}, P] = 0, \quad [\bar{D}_{\dot{\alpha}}^{\pm}, P] = 0, \quad [D^{\pm\pm}, P] = 0 \quad (1.5)$$

(see [5, 9] and Appendix A for details of the notation).

In the present paper, we study the QNS-deformation of $\mathcal{N} = (1, 1)$ supersymmetric $U(1)$ vector multiplet in harmonic superspace, thereby continuing the series of works on QS-deformations of the vector multiplet [9] and hypermultiplet [10]. To simplify things, we choose the deformation tensor in the particular factorizable form

$$\hat{C}_{ij}^{\alpha\beta} = b_{(ij)}c^{(\alpha\beta)}. \quad (1.6)$$

It breaks the “Lorentz” group $SU(2)_L$ and R-symmetry group $SU(2)$ down to $U(1)_L$ and $U(1)$, respectively. For this choice, we explicitly present the gauge transformation of the fields in the Wess-Zumino (WZ) gauge, as well as give a few examples of unbroken $\mathcal{N} = (1, 0)$ supersymmetry transformations. We also calculate the bosonic part of the deformed $\mathcal{N} = (1, 0)$ supersymmetric action. The choice (1.6) allows us to obtain *exact* expressions for the deformed $\mathcal{N} = (1, 0)$ gauge and supersymmetry transformation laws and the action, in contrast to the generic choice for which the corresponding expressions are known as formal series in the deformation parameters [7, 11]. More precisely, the authors of [7, 11] have obtained the all-order formula for the gauge and susy transformations in power-series form and explicitly presented the lowest orders. The unique possibility to deduce all results in closed form is a major reason why the non-singlet deformation with the ansatz (1.6) is worthy of a detailed study. When turning off some components of $\hat{C}_{ij}^{\alpha\beta}$

it is also possible to find exact expressions; see for example [12] where the exact gauge and $\mathcal{N} = (1, \frac{1}{2})$ supersymmetry transformations are obtained for a special case.

The bosonic action has a structure similar to the QS-deformed action calculated in [8, 9]. Namely, after a proper field redefinition the Lagrangian proves to differ from the undeformed one merely by a scalar factor which is a regular function of the argument

$$X = 2\bar{\phi}\sqrt{b^2c^2}, \quad \text{where } b^2 = b^{ij}b_{ij} \quad \text{and} \quad c^2 = c^{\alpha\beta}c_{\alpha\beta}, \quad (1.7)$$

and $\bar{\phi}$ is one of two real scalar fields of the gauge $N = (1, 1)$ multiplet. The precise relation is as follows

$$S = \int d^4x \cosh^2(2\bar{\phi}\sqrt{c^2b^2}) \left[-\frac{1}{2}\varphi\Box\bar{\phi} + \frac{1}{4}d^{ij}d_{ij} - \frac{1}{16}\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta} \right], \quad (1.8)$$

where $\tilde{F}_{\alpha\beta} = 2i\partial_{(\alpha\dot{\alpha}}\tilde{A}_{\beta)}^{\dot{\alpha}}$ is a self-dual component of the gauge field strength, with the potential $\tilde{A}_{\dot{\alpha}\beta}$ possessing the standard abelian gauge transformation law.² We observe that the only non-trivial irremovable interaction in the bosonic limit is that between $\bar{\phi}$ and $\tilde{F}_{\alpha\beta}$, as in the case of QS-deformation. Expanding our results in powers of X we show that the first-order terms are comparable with the formal series results obtained in [7, 11]. We also consider the case of degenerate QNS-deformation preserving the $\mathcal{N} = (1, \frac{1}{2})$ portion of supersymmetry [5]. It corresponds to the choice $b^2 = 0$. We find agreement with a recent paper [12] where the same option was treated within the $\mathcal{N} = 1$ superfield formalism.

The basic conventions used throughout the paper are summarized in Appendix A. Appendix B contains the list of useful harmonic integrals.

2 Generalities of Q-deformed $\mathcal{N} = (1, 1)$ gauge theory

2.1 The structure of generic Q-deformations

We use the harmonic superspace [23, 24] and work in its left-chiral basis, as defined in Appendix A, where

$$Q_{\alpha}^i = \partial_{\alpha}^i, \quad Q_{\alpha}^{\pm} \equiv Q_{\alpha}^i u_i^{\pm} = \pm \partial_{\mp\alpha}. \quad (2.1)$$

In this basis, the Poisson operator (1.1), (1.3) of the generic Q-deformation is written as

$$P = -\overleftarrow{\partial}_{\alpha}^i C_{ik}^{\alpha\beta} \overrightarrow{\partial}_{\beta}^k, \quad (2.2)$$

where

$$C_{ik}^{\alpha\beta} = \hat{C}_{(ik)}^{(\alpha\beta)} + I \epsilon^{\alpha\beta} \epsilon_{ik}. \quad (2.3)$$

²Recall that at the quadratic level $\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta}$ equals to the standard gauge field kinetic term $\sim \tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta} + \tilde{F}^{\dot{\alpha}\dot{\beta}}\tilde{F}_{\dot{\alpha}\dot{\beta}}$ modulo a total derivative.

Being expressed in terms of the harmonic projections

$$C^{\pm\pm\alpha\beta} = \hat{C}^{\pm\pm(\alpha\beta)}, \quad C^{\pm\mp\alpha\beta} = \hat{C}^{\pm\mp(\alpha\beta)} \pm I \epsilon^{\alpha\beta}, \quad \hat{C}^{+-}(\alpha\beta) = \hat{C}^{-+}(\alpha\beta), \quad (2.4)$$

the Poisson operator reads

$$\begin{aligned} P = & -\overleftarrow{\partial}_{+\alpha} \hat{C}^{++\alpha\beta} \overrightarrow{\partial}_{+\beta} - \overleftarrow{\partial}_{+\alpha} \left(\hat{C}^{+-\alpha\beta} + I \epsilon^{\alpha\beta} \right) \overrightarrow{\partial}_{-\beta} \\ & - \overleftarrow{\partial}_{-\alpha} \left(\hat{C}^{+-\alpha\beta} - I \epsilon^{\alpha\beta} \right) \overrightarrow{\partial}_{+\beta} - \overleftarrow{\partial}_{-\alpha} \hat{C}^{--\alpha\beta} \overrightarrow{\partial}_{-\beta}. \end{aligned} \quad (2.5)$$

The Q-deformed commutator of two Grassmann-even and harmonic-analytic superfields A and B can be calculated using the definition (1.2), (1.4) of the star product. For the superfields commuting with respect to the ordinary product, $[A, B] = 0$, the deformed commutator is given by

$$\begin{aligned} [A, B]_{\star} = & -2 \left[I \left(\partial_{-}^{\alpha} A \partial_{+\alpha} B - \partial_{+}^{\alpha} A \partial_{-\alpha} B \right) + (\partial_{-\alpha} A \partial_{-\beta} B) \hat{C}^{--\alpha\beta} \right. \\ & \left. + (\partial_{+\alpha} A \partial_{+\beta} B) \hat{C}^{++\alpha\beta} + (\partial_{-\alpha} A \partial_{+\beta} B + \partial_{+\alpha} A \partial_{-\beta} B) \hat{C}^{+-\alpha\beta} \right] \\ & - 3 [\partial_{-\alpha} (\partial_{+})^2 A \partial_{-\beta} (\partial_{+})^2 B] M^{++\alpha\beta}, \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} M^{++(\alpha\beta)} = & \hat{C}^{+-}(\alpha\gamma) \hat{C}_{(\gamma\mu)}^{++} \hat{C}^{+-}(\mu\beta) - \hat{C}^{++}(\alpha\gamma) \hat{C}_{(\gamma\mu)}^{++} \hat{C}^{--}(\mu\beta) \\ & - I \left[\hat{C}_{\gamma}^{++\alpha} \hat{C}^{+-}(\gamma\beta) + \hat{C}_{\gamma}^{++\beta} \hat{C}^{+-}(\gamma\alpha) \right] + I^2 \hat{C}^{++(\alpha\beta)}. \end{aligned} \quad (2.7)$$

Thus only first- and third-order terms contribute to the star commutator of the commuting analytic superfields. Since we are interested in the deformation of Abelian $\mathcal{N} = (1, 1)$ gauge theory, it will be basically sufficient to know the relations (2.6), (2.7). Note that for the special choice

$$\hat{C}_{(ik)}^{(\alpha\beta)} = c^{(\alpha\beta)} b_{(ik)} \quad (2.8)$$

the expression (2.7) drastically simplifies to

$$M^{++\alpha\beta} = c^{(\alpha\beta)} b^{++} \left(I^2 - \frac{1}{4} c^2 b^2 \right) \quad \text{with} \quad c^2 = c^{(\alpha\beta)} c_{(\alpha\beta)}, \quad b^2 = b^{(ik)} b_{(ik)}, \quad (2.9)$$

and vanishes for a particular relation between the deformation parameters, i.e.

$$I^2 = \frac{1}{4} c^2 b^2. \quad (2.10)$$

In what follows we shall use just the choice (2.8) in the component calculations since it allows one to obtain all the basic quantities in a closed form. While the generic $\hat{C}_{ik}^{\alpha\beta}$ fully

breaks both $SU(2)_L$ and $SU(2)$ automorphism symmetries, the particular ansatz (2.8) breaks these symmetries down to $U(1)_L$ and $U(1)$. Thus it is the maximally symmetric choice for the non-singlet deformation matrix. A generic matrix $\hat{C}_{ik}^{\alpha\beta}$ comprises three essential real parameters as compared to two such parameters in (2.8). In Sect. 4 we shall also consider a degenerate case of (2.8) with $b^2 = 0$.³

2.2 Deformed gauge transformations

The residual gauge transformations of the component fields of the Abelian $\mathcal{N} = (1, 1)$ vector multiplet in the WZ gauge can be found from the Q-deformed superfield transformation [9]

$$\delta_\Lambda V_{WZ}^{++} = D^{++}\Lambda + [V_{WZ}^{++}, \Lambda]_\star, \quad (2.11)$$

with V_{WZ}^{++} being the analytic harmonic $U(1)$ superfield gauge connection and Λ the analytic residual gauge parameter satisfying $D_\alpha^+\Lambda = \bar{D}_{\dot{\alpha}}^+\Lambda = 0$.

The superfield gauge parameter Λ should be chosen so as to preserve WZ gauge. In the left-chiral basis, where $x_A^{\alpha\dot{\alpha}} = x_L^{\alpha\dot{\alpha}} - 4i\theta^{-\alpha}\bar{\theta}^{+\dot{\alpha}}$ (see Appendix A for details), V_{WZ}^{++} has the following θ -expansion

$$V_{WZ}^{++} = v^{++} + \bar{\theta}_\alpha^+ v^{+\dot{\alpha}} + (\bar{\theta}^+)^2 v, \quad (2.12)$$

where

$$v^{++} = (\theta^+)^2 \bar{\phi}, \quad (2.13a)$$

$$v^{+\dot{\alpha}} = 2\theta^{+\alpha} A_\alpha^{\dot{\alpha}} + 4(\theta^+)^2 \bar{\Psi}^{-\dot{\alpha}} - 2i(\theta^+)^2 \theta^{-\alpha} \partial_\alpha^{\dot{\alpha}} \bar{\phi}, \quad (2.13b)$$

$$\begin{aligned} v = & \phi + 4\theta^+ \Psi^- + 3(\theta^+)^2 D^{--} - i(\theta^+ \theta^-) \partial^{\alpha\dot{\alpha}} A_{\alpha\dot{\alpha}} + \theta^{-\alpha} \theta^{+\beta} F_{\alpha\beta} \\ & - (\theta^+)^2 (\theta^-)^2 \square \bar{\phi} + 4i(\theta^+)^2 \theta^{-\alpha} \partial_{\alpha\dot{\alpha}} \bar{\Psi}^{-\dot{\alpha}}. \end{aligned} \quad (2.13c)$$

As the first step in deriving the gauge transformation of the components of V_{WZ}^{++} , we substitute in (2.11) the residual gauge parameter Λ_0 used in the undeformed and QS-deformed theories [9]. In chiral coordinates it reads

$$\Lambda_0 = ia + 2\theta^{-\alpha} \bar{\theta}^{+\dot{\alpha}} \partial_{\alpha\dot{\alpha}} a - i(\theta^-)^2 (\bar{\theta}^+)^2 \square a. \quad (2.14)$$

³The choice of $b^2 = 0$ is obviously inconsistent with the standard $SU(2)$ -covariant reality condition $(b_{ik})^\dagger = \varepsilon^{ij} \varepsilon^{kl} b_{jl}$ under which $b^2 = 0$ implies $b_{ik} = 0$. It is still compatible with a non-vanishing b_{ik} if the reality is defined with respect to a pseudo-conjugation and the R-symmetry group of $\mathcal{N} = (1, 1)$ superalgebra is $U(1)$ from the very beginning [5]. Fortunately, our further consideration does not depend on whether or not $c^{\alpha\beta}$ and b^{ik} are subject to any reality condition, which allows us not to care about this issue.

Next, we calculate the star-commutator involving only Λ_0 . Note that the third-order term in (2.6) does not contribute to the gauge transformation of V_{WZ}^{++} in view of the condition [9]

$$\partial_{+\alpha}\Lambda_0 = 0. \quad (2.15)$$

Explicit calculations of (2.11) with making use of (2.6), and (2.14) as the gauge parameter gives the following result

$$\begin{aligned} \delta_0 \bar{\phi} &= 0, \\ \delta_0 \bar{\Psi}_{\dot{\alpha}}^- &= 0, \\ \delta_0 A_{\alpha\dot{\alpha}} &= \partial_{\alpha\dot{\alpha}} a + 4 \bar{\phi} \partial_{\beta\dot{\alpha}} a \hat{C}_{\alpha}^{+-\beta}, \\ \delta_0 \phi &= 4 \partial_{\beta\dot{\beta}} a A_{\alpha}^{\dot{\beta}} \hat{C}^{+-\alpha\beta}, \\ \delta_0 D^{--} &= -\frac{4i}{3} \partial_{\beta\dot{\beta}} a \partial_{\alpha}^{\dot{\beta}} \bar{\phi} \hat{C}^{--\alpha\beta}, \\ \delta_0 \Psi^{-\alpha} &= -4 \bar{\Psi}^{-\dot{\gamma}} \partial_{\beta\dot{\gamma}} a \hat{C}^{+-\alpha\beta}. \end{aligned} \quad (2.16)$$

Here, we omitted the singlet terms ($\sim I$) which can be found in [9]. We observe that the transformations (2.16), in contrast to those of the singlet case, do not preserve WZ gauge because of the appearance of an unwanted dependence on the harmonic variables u_i^{\pm} in their right-hand sides. This forces us to choose the gauge parameter Λ as a sum of the ‘naïve’ one Λ_0 and some correction terms

$$\Lambda = \Lambda_0 + \Delta\Lambda, \quad (2.17)$$

with

$$\begin{aligned} \Delta\Lambda &= \theta_{\alpha}^+ \bar{\theta}_{\dot{\alpha}}^+ \partial_{\beta}^{\dot{\alpha}} a B_1^{--\alpha\beta} + (\bar{\theta}^+)^2 \partial_{\beta\dot{\beta}} a A_{\alpha}^{\dot{\beta}} G^{--\alpha\beta} + (\theta^+)^2 (\bar{\theta}^+)^2 \square a P^{-4} \\ &+ (\bar{\theta}^+)^2 \theta_{\alpha}^+ \left[\bar{\Psi}^{-\dot{\beta}} \partial_{\beta\dot{\beta}} a B_2^{--\alpha\beta} + \bar{\Psi}^{+\dot{\beta}} \partial_{\beta\dot{\beta}} a G^{-4\alpha\beta} \right] + (\theta^+)^2 (\bar{\theta}^+)^2 \partial_{\alpha\dot{\alpha}} a \partial_{\beta}^{\dot{\alpha}} \bar{\phi} B_3^{-4\alpha\beta} \\ &+ i \theta_{\alpha}^+ \theta_{\beta}^- (\bar{\theta}^+)^2 \square a B_1^{--\alpha\beta} + i \theta_{\alpha}^+ \theta_{\gamma}^- (\bar{\theta}^+)^2 \partial_{\beta\dot{\lambda}} a \partial^{\gamma\dot{\lambda}} \bar{\phi} \frac{d}{d\bar{\phi}} B_1^{--\alpha\beta}. \end{aligned} \quad (2.18)$$

The coefficients in (2.18) are some undetermined functions of harmonics, the field $\bar{\phi}$ and deformation parameters. Note that these coefficients involve both the symmetric and antisymmetric pieces in the spinor indices. Now we should calculate the correction term to $\delta_0 V_{\text{WZ}}^{++}$,

$$\hat{\delta} V_{\text{WZ}}^{++} = D^{++} \Delta\Lambda + [V_{\text{WZ}}^{++}, \Delta\Lambda]_{\star}. \quad (2.19)$$

From the structure of $\Delta\Lambda$ we can conclude that only the lowest order terms in the deformation parameters contribute to the star-commutator in (2.19) calculated according to (2.6) and that

the term $\sim \hat{C}^{--\alpha\beta}$ is vanishing. Thus we are left with

$$[V_{\text{WZ}}^{++}, \Delta\Lambda]_{\star} = -2I (\partial_{-}^{\alpha} V_{\text{WZ}}^{++} \partial_{+\alpha} \Delta\Lambda - \partial_{+}^{\alpha} V_{\text{WZ}}^{++} \partial_{-\alpha} \Delta\Lambda) - 2 (\partial_{+\alpha} V_{\text{WZ}}^{++} \partial_{+\beta} \Delta\Lambda) \hat{C}^{++\alpha\beta} \quad (2.20) \\ - 2 (\partial_{-\alpha} V_{\text{WZ}}^{++} \partial_{+\beta} \Delta\Lambda + \partial_{+\alpha} V_{\text{WZ}}^{++} \partial_{-\beta} \Delta\Lambda) \hat{C}^{+-\alpha\beta}.$$

Now we are ready to find the full gauge transformations of the fields following from

$$\delta V_{\text{WZ}}^{++} = \delta_0 V_{\text{WZ}}^{++} + \hat{\delta} V_{\text{WZ}}^{++}. \quad (2.21)$$

Requiring these full gauge transformations to preserve the WZ gauge amounts to the following conditions

$$\partial^{++} \delta A_{\alpha\dot{\alpha}} = 0 \leftrightarrow \partial^{--} \delta A_{\alpha\dot{\alpha}} = 0, \quad (2.22a)$$

$$\partial^{++} \delta \phi = 0 \leftrightarrow \partial^{--} \delta \phi = 0, \quad (2.22b)$$

$$(\partial^{++})^2 \delta \Psi_{\alpha}^{-} = 0 \leftrightarrow \partial^{--} \delta \Psi_{\alpha}^{-} = 0, \quad (2.22c)$$

$$(\partial^{++})^3 \delta D^{--} = 0 \leftrightarrow \partial^{--} \delta D^{--} = 0. \quad (2.22d)$$

After substituting the precise form of the gauge variations, these conditions fix the unknown harmonic functions in terms of $\bar{\phi}$, deformation parameters and harmonics. After solving them, one can find the explicit form of the gauge variations. Unfortunately, it is very difficult to find the closed solution of these equations for the generic deformation parameters, though their perturbative solution always exists as an infinite series in these parameters.

Remarkably, the solution can be found in a closed form if one assumes the product structure (2.8) for the non-singlet part of the deformation matrix. In the rest of our paper we will deal just with this choice, though its possible stringy origin, e.g. as some special $\mathcal{N} = 4$ superstring background,⁴ still remains to be revealed.

3 The precise form of the residual gauge transformations

In the first two subsections we present, as instructive examples, the calculation of $\delta A_{\alpha\dot{\alpha}}$ and $\delta \phi$ for the choice (2.8). The full list of the residual gauge transformations is given in the last subsection.

⁴The stringy interpretation of the QS-deformation was given in [9].

3.1 Gauge transformation of $A_{\alpha\dot{\alpha}}$

The full expression for the variation of $A_{\alpha\dot{\alpha}}$ following from (2.21) upon using (2.18) is

$$\delta A_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} a + 4\partial_{\beta\dot{\alpha}} a \bar{\phi} \hat{C}_{\alpha}^{+-\beta} + 2\partial_{\beta\dot{\alpha}} a \bar{\phi} B_{1\rho}^{--\beta} \hat{C}_{\alpha}^{++\rho} + \frac{1}{2} \partial_{\beta\dot{\alpha}} a \partial^{++} B_{1\alpha}^{--\beta}. \quad (3.1)$$

The condition (2.22a) amounts to the harmonic equation

$$(\partial^{++})^2 B_{1\alpha}^{--\beta} + 4\bar{\phi} \hat{C}_{\alpha}^{++\rho} \partial^{++} B_{1\rho}^{--\beta} + 8\bar{\phi} \hat{C}_{\alpha}^{++\beta} = 0, \quad (3.2)$$

which is equivalent to the system

$$(\partial^{++})^2 B^{--} - 2\bar{\phi} \hat{C}^{++\alpha\beta} \partial^{++} \hat{B}_{\alpha\beta}^{--} = 0, \quad (3.3a)$$

$$(\partial^{++})^2 \hat{B}^{--\alpha\beta} + 4\bar{\phi} \hat{C}^{++\alpha\beta} \partial^{++} B^{--} + 4\bar{\phi} \hat{C}^{++(\alpha\rho} \partial^{++} \hat{B}_{1\rho}^{--\beta)} + 8\bar{\phi} \hat{C}^{++\alpha\beta} = 0. \quad (3.3b)$$

Here

$$B_1^{--\alpha}{}_{\beta} \equiv \hat{B}^{--\alpha}{}_{\beta} + \delta_{\beta}^{\alpha} B^{--}, \quad \hat{B}^{--\alpha}{}_{\alpha} = 0. \quad (3.4)$$

Though it is easy to obtain a closed equation for

$$\mathcal{G} = \partial^{++} B^{--} + 2, \quad (3.5)$$

viz.

$$(\partial^{++})^2 \mathcal{G} + 8\bar{\phi}^2 (\hat{C}^{++})^2 \mathcal{G} = 0, \quad (\hat{C}^{++})^2 \equiv \hat{C}^{++\alpha\beta} \hat{C}^{++}{}_{\alpha\beta}, \quad (3.6)$$

its solution in a close form is very difficult to find without further simplifications. Equations (3.3a), (3.6), for example, can be solved by iterations to any order in the deformation parameter. Closed solutions for these and the remaining constraints in (2.22) can be found in the simplified case (2.8). For this case we can expand tensor fields over the basis $\{c^{\alpha\beta}, \varepsilon^{\alpha\beta}\}$, for instance

$$B_1^{--\alpha\beta} = F^{--} c^{\alpha\beta} + B^{--} \varepsilon^{\alpha\beta}. \quad (3.7)$$

Then, defining

$$\mathcal{F} \equiv \partial^{++} F^{--}, \quad (3.8)$$

as a consequence of eqs. (3.3a) we obtain the following equations

$$\partial^{++} \mathcal{G} - 2\bar{\phi} c^2 b^{++} \mathcal{F} = 0, \quad (3.9a)$$

$$\partial^{++} \mathcal{F} + 4\bar{\phi} b^{++} \mathcal{G} = 0, \quad (3.9b)$$

where $c^2 = c^{\alpha\beta} c_{\alpha\beta}$. Applying ∂^{++} to (3.9) once more, we arrive at the decoupled system

$$\begin{aligned} (a) \quad & (\partial^{++})^2 \mathcal{G} + (\kappa^{++})^2 \mathcal{G} = 0, \\ (b) \quad & (\partial^{++})^2 \mathcal{F} + (\kappa^{++})^2 \mathcal{F} = 0, \end{aligned} \quad (3.10)$$

where $\kappa^{++} = 2\bar{\phi}\sqrt{2c^2} b^{+-}$. These equations are a sort of the harmonic oscillator ones and they are solved by

$$\mathcal{F} = (C_1, C_2), \quad \mathcal{G} = (C_3, C_4), \quad (3.11)$$

where

$$(C_i, C_j) \equiv C_i \cos Z + C_j \sin Z, \quad (3.12)$$

C_1, C_2, C_3 and C_4 are complex integration constants and

$$Z = 2\bar{\phi}\sqrt{2c^2} b^{+-} \quad (3.13)$$

(note that $\kappa^{++} = \partial^{++} Z$). Substituting (3.11) into (3.9) we find two relations between the integration constants

$$C_3 = -\frac{1}{2} \sqrt{2c^2} C_2, \quad C_4 = \frac{1}{2} \sqrt{2c^2} C_1. \quad (3.14)$$

Taking into account the definition (3.5) and (3.8), as well as the property that harmonic integrals of the full harmonic derivatives are vanishing, we also find

$$\int du (C_1, C_2) = 0, \quad \int du (-C_2, C_1) = \frac{4}{\sqrt{2c^2}}. \quad (3.15)$$

To compute these integrals, we start by noting that

$$b^{++} b^{--} - (b^{+-})^2 = \lambda \equiv \frac{1}{2} b^2, \quad b^2 = b^{ik} b_{ik}. \quad (3.16)$$

It is then easy to show that

$$(b^{+-})^{2k+1} = \partial^{++} \xi^{--}, \quad (b^{+-})^{2k} = \partial^{++} \chi^{--} + (-1)^k \frac{1}{2k+1} \lambda^k, \quad (3.17)$$

where ξ^{--} and χ^{--} are some harmonic functions whose precise form is of no relevance for computing integrals (3.15). From eqs. (3.17) it follows that the harmonic integral of any odd power of b^{+-} is vanishing, whence, e.g., $\int du \sin Z = 0$. In Appendix B we give a list of relevant non-vanishing harmonic integrals. Consulting it, we solve for the constants in (3.15)

$$C_1 = 0, \quad C_2 = -\frac{4}{\sqrt{2c^2}} \frac{X}{\sinh X}, \quad (3.18)$$

where

$$X = 2\bar{\phi}\sqrt{c^2b^2}. \quad (3.19)$$

The final form of the solution for \mathcal{F} and \mathcal{G} is

$$\mathcal{F} = \partial^{++}F^{--} = -\frac{4}{\sqrt{2c^2}}\left(0, \frac{X}{\sinh X}\right), \quad \mathcal{G} = \partial^{++}B^{--} + 2 = -2\left(\frac{X}{\sinh X}, 0\right), \quad (3.20)$$

For our purpose of finding the closed form of $\delta A_{\alpha\dot{\alpha}}$ there is no need to compute B^{--} and F^{--} . Indeed, the coefficient of $\partial_{\alpha\dot{\alpha}}a$ in (3.1),

$$L_{\beta}^{\alpha} = \delta_{\beta}^{\alpha} (1 - \bar{\phi}c^2 F^{--}b^{++}) + b^{+-}c_{\beta}^{\alpha} + 2c_{\beta}^{\alpha}\bar{\phi}B^{--}b^{++} + \frac{1}{2}\partial^{++}B_{1\beta}^{--\alpha}, \quad (3.21)$$

is required not to depend on harmonics. Multiplying (3.21) by $\int du 1 = 1$ and integrating by parts with respect to the harmonic derivative ∂^{++} , we can represent this coefficient as the following harmonic integral

$$L_{\beta}^{\alpha} = \int du [\delta_{\beta}^{\alpha} (1 + 2\bar{\phi}^2 c^2 \mathcal{G} b^{++}b^{--}) + 2c_{\beta}^{\alpha}\bar{\phi}^2 c^2 \mathcal{F} b^{++}b^{--}]. \quad (3.22)$$

Keeping in mind (3.16) and the first of eqs. (3.17), the traceless part in (3.22) can be shown to vanish and the computation of L_{β}^{α} is reduced to computing the harmonic integral

$$J = \int du (b^{++}b^{--})\mathcal{G}. \quad (3.23)$$

This is easy to perform using (3.16) and the formulas of Appendix B:

$$J = \frac{1}{2\bar{\phi}^2 c^2} (X \coth X - 1) = 2b^2 \left(\frac{X \coth X - 1}{X} \right). \quad (3.24)$$

Finally, we obtain

$$L_{\beta}^{\alpha} = \delta_{\beta}^{\alpha} X \coth X. \quad (3.25)$$

Thus the variation $\delta A_{\alpha\dot{\alpha}}$ proves to have the very simple form

$$\delta A_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}a X \coth X. \quad (3.26)$$

Note that $X \coth X$ is a well behaved function having the proper undeformed limit $X \longrightarrow 0$,

$$\delta A_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}a \left(1 + \frac{X^2}{3} - \frac{X^4}{45} + \dots \right). \quad (3.27)$$

3.2 Gauge transformation of ϕ

The full gauge variation of ϕ is a sum of $\delta_0\phi$, eq. (2.16), and the part computed from (2.21) with Λ being substituted by $\Delta\Lambda$ (2.18). We have

$$\delta\phi = \partial_{\alpha\dot{\alpha}} a A_{\dot{\beta}}^{\dot{\alpha}} \left[\partial^{++} G^{--\beta\alpha} + 4\hat{C}^{+-\beta\alpha} + 2 B_{1\rho}^{--\alpha} \hat{C}^{++\rho\beta} \right] \equiv \partial_{\alpha\dot{\alpha}} a A_{\dot{\beta}}^{\dot{\alpha}} G^{\beta\alpha}. \quad (3.28)$$

The r.h.s. of this transformation must be independent of harmonics as stated in (2.22b). This amounts to the condition

$$(\partial^{++})^2 G_{\beta}^{--\alpha} + 2\hat{C}_{\beta}^{++\rho} \partial^{++} B_{1\rho}^{--\alpha} = 0. \quad (3.29)$$

It can be easily solved for the unknown $G^{--\beta\alpha}$. However, it is not necessary to know this solution for finding $\delta\phi$, since

$$\delta\phi = \partial_{\alpha\dot{\alpha}} a A_{\dot{\beta}}^{\dot{\alpha}} \int du G^{\beta\alpha} \quad (3.30)$$

and the problem is reduced to the computation of the harmonic integral in (3.30). For the product ansatz (2.8) this integral, modulo a total harmonic derivative in the integrand, is given by

$$\int du G^{\beta\alpha} = - \int du \left(\epsilon^{\alpha\beta} c^2 b^{+-} \mathcal{F} + 2 c^{(\alpha\beta)} b^{+-} \mathcal{G} \right). \quad (3.31)$$

The second term under the integral in (3.31) is a total harmonic derivative in virtue of the first relation in (3.17) and is therefore vanishing, whence

$$\delta\phi = 2 \sqrt{c^2 b^2} (\partial_{\alpha\dot{\alpha}} a A^{\alpha\dot{\alpha}}) \frac{1 - X \coth X}{X}. \quad (3.32)$$

Note that the r.h.s. of (3.32) contains only integer powers of $c^2 b^2$, as can be seen from the definition (3.19) and the fact that the function of X in (3.32) contains only odd powers of X .

3.3 The full set of QNS-deformed gauge variations

Proceeding in the same way as above, one can find the full set of QNS-deformed gauge transformations laws for the $\mathcal{N} = (1, 1)$ vector multiplet in WZ gauge. Here is the list of them:

$$\delta \bar{\phi} = 0, \quad \delta \bar{\Psi}_\alpha^k = 0, \quad (3.33)$$

$$\delta A_{\alpha\dot{\alpha}} = X \coth X \partial_{\alpha\dot{\alpha}} a, \quad (3.34)$$

$$\delta \phi = 2\sqrt{c^2 b^2} \left(\frac{1 - X \coth X}{X} \right) A^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} a, \quad (3.35)$$

$$\begin{aligned} \delta \Psi_\alpha^i = & \left\{ \left[\frac{4X^2(X \coth X - 1)}{X^2 + \sinh^2 X - X \sinh 2X} \right] b^{ij} c_{\alpha\beta} \right. \\ & \left. - \sqrt{c^2 b^2} \left[\frac{4X \cosh^2 X - 2X^2(\coth X + X) - \sinh 2X}{X^2 + \sinh^2 X - X \sinh 2X} \right] \varepsilon^{ij} \varepsilon_{\alpha\beta} \right\} \bar{\Psi}_{j\dot{\alpha}} \partial^{\beta\dot{\alpha}} a. \end{aligned} \quad (3.36)$$

$$\delta D_{ij} = 2ib_{ij} c^{\alpha\beta} \partial_{\alpha\dot{\alpha}} \bar{\phi} \partial_\beta^{\dot{\alpha}} a. \quad (3.37)$$

Having the explicit form of the deformed gauge transformations, one can deduce a minimal Seiberg-Witten-like map (SW map) which puts these transformations into the standard undeformed form

$$A_{\alpha\dot{\alpha}} = \tilde{A}_{\alpha\dot{\alpha}} X \coth X, \quad (3.38)$$

$$\phi = \tilde{\phi} + \tilde{A}^2 \sqrt{c^2 b^2} X \coth X \left(\frac{1 - X \coth X}{X} \right), \quad (3.39)$$

$$\begin{aligned} \Psi^{i\alpha} = & \tilde{\Psi}^{i\alpha} + \left\{ \left[\frac{4X^2(X \coth X - 1)}{X^2 + \sinh^2 X - X \sinh 2X} \right] b^{ij} c_{\alpha\beta} \right. \\ & \left. - \sqrt{c^2 b^2} \left[\frac{4X \cosh^2 X - 2X^2(\coth X + X) - \sinh 2X}{X^2 + \sinh^2 X - X \sinh 2X} \right] \varepsilon^{ij} \varepsilon_{\alpha\beta} \right\} \bar{\Psi}_{j\dot{\alpha}} \tilde{A}^{\beta\dot{\alpha}}. \end{aligned} \quad (3.40)$$

$$D_{ij} = \tilde{D}_{ij} + 2ib_{ij} c^{\alpha\beta} \partial_{\alpha\dot{\alpha}} \bar{\phi} \tilde{A}_\beta^{\dot{\alpha}}. \quad (3.41)$$

For the fields with “tilde” we obtain the standard transformations

$$\delta \tilde{A}_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} a, \quad \delta \tilde{\phi} = 0, \quad \delta \tilde{D}^{ij} = 0, \quad \delta \tilde{\Psi}_\alpha^k = 0. \quad (3.42)$$

The gauge field strength $F_{\alpha\beta} = 2i\partial_{(\alpha\dot{\alpha}} A_{\beta)}^{\dot{\alpha}}$ which is non-covariant with respect to the deformed transformations is redefined under the transformation $A_{\alpha\dot{\alpha}} \rightarrow \tilde{A}_{\alpha\dot{\alpha}}$ as

$$F_{\alpha\beta} = \tilde{F}_{\alpha\beta} X \coth X + 4i\sqrt{b^2 c^2} \tilde{A}_{(\beta\dot{\alpha}} \partial_{\alpha)}^{\dot{\alpha}} \bar{\phi} \left(\coth X - \frac{X}{\sinh^2 X} \right), \quad (3.43)$$

where $\tilde{F}_{\alpha\beta} = 2i\partial_{(\alpha\dot{\alpha}}\tilde{A}_{\beta)}^{\dot{\alpha}}$. Since $\tilde{F}_{\alpha\beta}$ is manifestly gauge invariant, it is easy to derive from (3.43) what is the deformed analog of this field strength.

4 The QNS-deformed action

In this section we calculate the $\mathcal{N} = (1, 0)$ gauge invariant action in components. We concentrate on the bosonic limit of the action. The full supersymmetric action with all the fermionic fields included, as well as the full set of unbroken $\mathcal{N} = (1, 0)$ supersymmetry transformation laws,⁵ will be presented in a forthcoming paper [25].

The QNS-deformed action for the $\mathcal{N} = (1, 1)$ U(1) gauge theory in harmonic superspace [24], in the form most appropriate for our purposes, is written in the same way as in the QS-deformed case [9]

$$S = \frac{1}{4} \int d^4x_L d^4\theta du \mathcal{W} \star \mathcal{W} = \frac{1}{4} \int d^4x d^4\theta du \mathcal{W}^2. \quad (4.1)$$

Here \mathcal{W} is the covariant superfield strength

$$\mathcal{W} = -\frac{1}{4}(\bar{D}^+)^2 V^{--} \equiv \mathcal{A} + \bar{\theta}_{\dot{\alpha}}^+ \tau^{-\dot{\alpha}} + (\bar{\theta}^+)^2 \tau^{--}, \quad (4.2)$$

and V^{--} is the non-analytic harmonic connection related to V_{WZ}^{++} by the harmonic flatness condition

$$D^{++}V^{--} - D^{--}V_{\text{WZ}}^{++} + [V_{\text{WZ}}^{++}, V^{--}]_{\star} = 0. \quad (4.3)$$

In (4.2) we have used the general expansion of V^{--} in terms of chiral superfield components (depending only on $x_L^{\alpha\dot{\alpha}}, \theta_{\alpha}^{\pm}, u^{\pm i}$)

$$\begin{aligned} V^{--} = & v^{--} + \bar{\theta}_{\dot{\alpha}}^+ v^{(-3)\dot{\alpha}} + \bar{\theta}_{\dot{\alpha}}^- v^{-\dot{\alpha}} + (\bar{\theta}^-)^2 \mathcal{A} + (\bar{\theta}^+ \bar{\theta}^-) \varphi^{--} + \bar{\theta}^{-\dot{\alpha}} \bar{\theta}^{+\dot{\beta}} \varphi_{\dot{\alpha}\dot{\beta}}^{--} + (\bar{\theta}^+)^2 v^{(-4)} \\ & + (\bar{\theta}^-)^2 \bar{\theta}_{\dot{\alpha}}^+ \tau^{-\dot{\alpha}} + (\bar{\theta}^+)^2 \bar{\theta}_{\dot{\alpha}}^- \tau^{(-3)\dot{\alpha}} + (\bar{\theta}^+)^2 (\bar{\theta}^-)^2 \tau^{--}. \end{aligned} \quad (4.4)$$

The whole effect of the considered deformation in the above action comes from the structure of \mathcal{W} due to the presence of the star commutator in the equation (4.3) defining V^{--} . As a consequence of the latter, (4.2) satisfies the condition

$$D^{++}\mathcal{W} + [V_{\text{WZ}}^{++}, \mathcal{W}]_{\star} = 0, \quad (4.5)$$

⁵In fact, it is of no actual necessity to explicitly know these transformations, since our procedure of deriving the action is manifestly $\mathcal{N} = (1, 0)$ supersymmetric by construction.

which amounts to the following equations for the chiral coefficients in (4.2)

$$\nabla^{++}\mathcal{A} = 0, \quad (4.6a)$$

$$\nabla^{++}\tau^{-\dot{\alpha}} + [v^{+\dot{\alpha}}, \mathcal{A}]_{\star} = 0, \quad (4.6b)$$

$$\nabla^{++}\tau^{--} - \frac{1}{2} \{v_{\dot{\alpha}}^{+}, \tau^{-\dot{\alpha}}\}_{\star} + [v, \mathcal{A}]_{\star} = 0, \quad (4.6c)$$

where

$$\nabla^{++} = D^{++} + [v^{++},]_{\star} \quad (4.7)$$

and $v^{++} = (\theta^{+})^2 \bar{\phi}$, $v_{\dot{\alpha}}^{+}$ and v are defined in (2.13b), (2.13c). The Q-deformed commutator in (4.7), for a general chiral superfield $\Phi(x_L, \theta_{\alpha}^{\pm})$ (irrespective of the Grassmann parity of the latter), reads

$$[v^{++}, \Phi]_{\star} = -2\partial_{+\alpha}v^{++}\partial_{+\beta}\Phi\hat{C}^{++\alpha\beta} - 2\partial_{+\alpha}v^{++}\partial_{-\beta}\Phi\hat{C}^{+-\alpha\beta} - 2I\partial_{+\alpha}v^{++}\partial_{-\beta}\Phi\varepsilon^{\alpha\beta}. \quad (4.8)$$

Then for the product ansatz (2.8), $\nabla^{++}\Phi$ becomes

$$\nabla^{++}\Phi = \left[\partial^{++} - \left(\varepsilon^{\alpha\beta} + 4\bar{\phi}b^{+-}c^{\alpha\beta} \right) \theta_{\alpha}^{+}\partial_{-\beta} - 4\bar{\phi}b^{++}c^{\alpha\beta}\theta_{\alpha}^{+}\partial_{+\beta} \right] \Phi. \quad (4.9)$$

Like in the QS-case [9], using eqs. (4.6b), (4.6c) and the following additional equations implied by (4.3) for the other chiral coefficients in the expansion (4.4)

$$\nabla^{++}v^{-\dot{\alpha}} - v^{+\dot{\alpha}} = 0, \quad (4.10a)$$

$$\nabla^{++}\varphi^{--} + 2(\mathcal{A} - v) + \frac{1}{2} \{v^{+\dot{\alpha}}, v_{\dot{\alpha}}^{-}\}_{\star} = 0, \quad (4.10b)$$

it is straightforward to find the explicit form of $\tau^{-\dot{\alpha}}, \tau^{--}$ and to show that the only superfield that contributes to the action is \mathcal{A} . Thus the invariant action is reduced to

$$S = \frac{1}{4} \int d^4x d^4\theta du \mathcal{A}^2, \quad (4.11)$$

and it remains to calculate the superfield \mathcal{A} . This can be accomplished using eqs. (4.6a), (4.10a) and (4.10b). As a first step, we substitute in (4.11) the component expansion of \mathcal{A}

$$\begin{aligned} \mathcal{A} = & \mathcal{A}_1 + \theta^{-\alpha}\mathcal{A}_{2\alpha}^{+} + \theta^{+\alpha}\mathcal{A}_{3\alpha}^{-} + (\theta^{-})^2\mathcal{A}_4^{++} + (\theta^{-}\theta^{+})\mathcal{A}_5 + \theta^{-\alpha}\theta^{+\beta}\mathcal{A}_{6\alpha\beta} + (\theta^{+})^2\mathcal{A}_7^{--} \\ & + (\theta^{-})^2\theta^{+\alpha}\mathcal{A}_{8\alpha}^{+} + (\theta^{+})^2\theta^{-\alpha}\mathcal{A}_{9\alpha}^{-} + (\theta^{-})^2(\theta^{+})^2\mathcal{A}_{10}, \end{aligned} \quad (4.12)$$

and integrate over Grassmann coordinates, which leads to

$$S = \frac{1}{4} \int d^4 x_L du \left[2\mathcal{A}_1 \mathcal{A}_{10} - (\mathcal{A}_2^+ \mathcal{A}_9^-) - (\mathcal{A}_3^- \mathcal{A}_8^+) + 2\mathcal{A}_4^{++} \mathcal{A}_7^{--} - \frac{1}{2} \mathcal{A}_5^2 - \frac{1}{4} \mathcal{A}_6^2 \right]. \quad (4.13)$$

As soon as we limit ourselves to the bosonic part of the action, the terms $\mathcal{A}_{2\alpha}^+, \mathcal{A}_{3\alpha}^-, \mathcal{A}_{8\alpha}^+$ and $\mathcal{A}_{9\alpha}^-$ can be discarded. Thus the bosonic action we are searching for takes the form

$$S_{bos} = \frac{1}{4} \int d^4 x_L du \left[2\mathcal{A}_1 \mathcal{A}_{10} + 2\mathcal{A}_4^{++} \mathcal{A}_7^{--} - \frac{1}{2} \mathcal{A}_5^2 - \frac{1}{4} \mathcal{A}_6^2 \right]. \quad (4.14)$$

4.1 Determining \mathcal{A}

By substituting the θ expansion (4.12), with all fermionic components omitted, into the superfield equation (4.6a), we obtain the following equations for the quantities entering the bosonic action (4.14):

$$\partial^{++} \mathcal{A}_1 = 0, \quad (4.15a)$$

$$\partial^{++} \mathcal{A}_4^{++} = 0, \quad (4.15b)$$

$$\partial^{++} \mathcal{A}_5 + 2\mathcal{A}_4^{++} - 2\bar{\phi} b^{++} c^{\alpha\beta} \mathcal{A}_{6\alpha\beta} = 0, \quad (4.15c)$$

$$\partial^{++} \mathcal{A}_{6\alpha\beta} + 4\bar{\phi}(2b^{+-} \mathcal{A}_4^{++} + b^{++} \mathcal{A}_5) c_{\alpha\beta} + 4b^{++} \bar{\phi} \mathcal{A}_{6(\alpha\gamma} c_{\beta)}^\gamma = 0, \quad (4.15d)$$

$$\partial^{++} \mathcal{A}_7^{--} + \mathcal{A}_5 + 2\bar{\phi} b^{+-} c^{\alpha\beta} \mathcal{A}_{6\alpha\beta} = 0, \quad (4.15e)$$

$$\partial^{++} \mathcal{A}_{10} = 0. \quad (4.15f)$$

It immediately follows from (4.15a), (4.15b) and (4.15f) that \mathcal{A}_1 and \mathcal{A}_{10} are independent of harmonics and that \mathcal{A}_4^{++} is of the form $\mathcal{A}_4^{++} = \mathcal{A}_4^{ij} u_i^+ u_j^+$, with \mathcal{A}_4^{ij} being independent of harmonics. It is also obvious that the equations (4.15), being homogeneous, can determine the components of \mathcal{A} only up to some integration constants. These constants should be fixed from eqs. (4.10b) and (4.10a). To accomplish this, we first have to pass to components in $v^{-\dot{\alpha}}$ and φ^{--} as in (4.12) and substitute the relevant θ expansions (with all fermions omitted) into eqs. (4.10b) and (4.10a). After that we should solve the corresponding harmonic equations for the components. Omitting details, we obtain the following solutions for \mathcal{A}_1 and \mathcal{A}_{10} :

$$\begin{aligned} \mathcal{A}_1 &= \phi + \frac{1}{2} \bar{\phi}^{-1} \left(1 - \frac{\tanh X}{X} \right) A^2 + (b^2 c^2)^{3/2} \tanh X \partial_{\alpha\dot{\alpha}} \bar{\phi} \partial^{\alpha\dot{\alpha}} \bar{\phi}, \\ \mathcal{A}_{10} &= -\square \bar{\phi}, \quad A^2 \equiv A^{\alpha\dot{\alpha}} A_{\alpha\dot{\alpha}}. \end{aligned} \quad (4.16)$$

Solving the rest of eqs. (4.15) is not so easy, though things are simplified by observing that we do not need to calculate \mathcal{A}_7^{--} . Indeed, making use of the relation $\mathcal{A}_4^{++} = \partial^{++}\mathcal{A}_4^{+-}$, $\mathcal{A}_4^{+-} = \mathcal{A}_4^{(ij)}u_i^+u_j^-$ and integrating by parts in the term

$$\int d^4x du \frac{1}{2} \mathcal{A}_4^{++} \mathcal{A}_7^{--} = - \int d^4x du \frac{1}{2} \mathcal{A}_4^{+-} \partial^{++} \mathcal{A}_7^{--}, \quad (4.17)$$

in (4.14), we can eliminate $\partial^{++}\mathcal{A}_7^{--}$ with the help of eq. (4.15e). As a result, the remaining part of the action will involve only \mathcal{A}_4^{ij} , \mathcal{A}_5 and $\mathcal{A}_{6\alpha\beta}$. Our strategy for finding explicit expressions for these quantities as solutions of the appropriate component harmonic equations is as follows. Firstly we find series solutions to few first orders in the deformation parameters, with taking account of the component equations comprised by (4.10b). Such a perturbative solution suggests a particular ansatz for the sought component fields, which finally provides an exact solution reproducing the known terms in the series solution. In this way we arrive at the following ansatz

$$\begin{aligned} \mathcal{A}_{6\alpha\beta} &= g_1 F_{\alpha\beta} + g_2 c_{\alpha\beta} + g_3 c_{(\alpha}^\gamma F_{\gamma\beta)} + g_4 A_{(\alpha\dot{\alpha}} \partial_{\beta)}^{\dot{\alpha}} \bar{\phi} + g_5 c_{(\alpha}^\gamma A_{(\gamma\dot{\alpha}} \partial_{\beta)}^{\dot{\alpha}} \bar{\phi}, \\ \varphi_{6\alpha\beta}^{--} &= h_1^{--} F_{\alpha\beta} + h_2^{--} c_{\alpha\beta} + h_3^{--} c_{(\alpha}^\gamma F_{\gamma\beta)} + h_4^{--} A_{(\alpha\dot{\alpha}} \partial_{\beta)}^{\dot{\alpha}} \bar{\phi} + h_5^{--} c_{(\alpha}^\gamma A_{(\gamma\dot{\alpha}} \partial_{\beta)}^{\dot{\alpha}} \bar{\phi}, \\ \mathcal{A}_4^{ij} &= \alpha_1 D^{ij} + \alpha_2 b^{ij} + \alpha_2 b^{(ik} D_k^{j)}. \end{aligned} \quad (4.18)$$

The functions g_i , h_i^{--} can depend on $\bar{\phi}$, b^{ij} , $c_{\alpha\beta}$ and harmonics. Similarly, α_i can depend only on $\bar{\phi}$, b^{ij} and $c_{\alpha\beta}$; these functions are harmonic- independent. To find all these functions one actually needs eq. (4.15d) (which amounts to several coupled equations after substituting the above ansatz for $\mathcal{A}_{6\alpha\beta}$) and those component equations which appear as the coefficients of the monomials $(\theta^-)^2$, $(\theta^{-\alpha}\theta^{+\beta})$ and $(\theta^+)^2$ in (4.10b). Once the unknowns in (4.18) are found, the function \mathcal{A}_5 can be computed from (4.15c), without assuming beforehand any ansatz for it. Skipping details of calculations and introducing the short-hand notation

$$\begin{aligned} c \cdot F &= c^{\alpha\beta} F_{\alpha\beta}, \quad b \cdot D = b_{ij} D^{ij}, \quad c \cdot A \partial \bar{\phi} = c^{\alpha\beta} A_{(\alpha\dot{\alpha}} \partial_{\beta)}^{\dot{\alpha}} \bar{\phi}, \\ A \cdot \partial \bar{\phi} &= A_{\alpha\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \bar{\phi}, \quad F \cdot A \partial \bar{\phi} = F^{\alpha\beta} A_{(\alpha\dot{\alpha}} \partial_{\beta)}^{\dot{\alpha}} \bar{\phi}, \end{aligned}$$

the final expressions for the remaining building blocks of the bosonic action are

$$\mathcal{A}_4^{ij} = \frac{\sigma b^{ij}}{\cosh^3 X} + \frac{D^{ij}}{\cosh^2 X}, \quad (4.19)$$

$$\begin{aligned} \mathcal{A}_5 = & -\frac{2\sigma b^{+-}}{\cosh^3 X} + \frac{\sqrt{2b^2}}{X^2 \cosh^3 X} \left(0, -2i \cosh^2 X (\cosh X \sinh X - 1)(c \cdot A \partial \bar{\phi}) \right. \\ & \left. - X^2 \sinh X \sigma + \bar{\phi} \cosh^3 X \sinh X (c \cdot F) \right) + \frac{D^{+-}}{\cosh X} (-2, 0) \\ & + \frac{\bar{\phi} \sqrt{2c^2} b^{--} D^{++}}{(X^2 + Z^2)} [(0, -X \sinh X) - Z + (\cosh X, 0)Z], \end{aligned} \quad (4.20)$$

$$g_1 = \left(\frac{\sinh X}{X}, 0 \right), \quad (4.21a)$$

$$\begin{aligned} g_2 = & -\frac{2\bar{\phi} b^2 [(X^2 + Z^2)\sigma + 8\bar{\phi}^2 c^2 \cosh X b^{--} D^{++}]}{X^2 \cosh^3 X (X^2 + Z^2)} [(0, 2 \cosh X)Z + (2X \sinh X, 0)] \\ & + \sqrt{\frac{2}{c^2}} \left(0, \frac{2D^{+-}}{\cosh X} + \frac{2b^{+-}\sigma}{\cosh^2 X} \right), \end{aligned} \quad (4.21b)$$

$$g_3 = -\sqrt{\frac{2}{c^2}} \left(0, \frac{\sinh X}{X} \right), \quad (4.21c)$$

$$g_4 = \frac{2i\sqrt{b^2 c^2}}{X^2 \cosh X} (X - \cosh X \sinh X, 0), \quad (4.21d)$$

$$g_5 = \frac{2i\sqrt{b^2 c^2}}{X^2 \cosh X} (0, \cosh X \sinh X - X). \quad (4.21e)$$

Here

$$\sigma = \frac{\sinh X}{X} \left[-2i(c \cdot A \partial \bar{\phi}) \frac{\cosh X \sinh X - 2X}{X} + (c \cdot F) \bar{\phi} \frac{\cosh X \sinh X}{X} \right] \quad (4.22)$$

and we used the definition (3.12).

All these expressions meet the criterion of regularity in b^2 , c^2 and $\bar{\phi}$, as can be checked on their closer inspection. In particular, functions multiplied by $1/(X^2 + Z^2)$ in the expressions for \mathcal{A}_5 and g_2 fulfill the consistency and regularity conditions which mean that they are in fact of the form $(X^2 + Z^2)F(X, Z)$, where $F(X, Z)$ is regular. This regularity can be seen most clearly

by representing

$$\begin{aligned} & \frac{1}{X^2 + Z^2} [(0, -X \sinh X) - Z + (\cosh X, 0)Z] \\ &= \frac{i}{2} \left(\frac{\cosh(X + iZ) - 1}{X + iZ} - \frac{\cosh(X - iZ) - 1}{X - iZ} \right), \\ & \frac{1}{X^2 + Z^2} [(2X \sinh X, 0) + (0, 2 \cosh X)Z] = \frac{\sinh(X + iZ)}{X + iZ} + \frac{\sinh(X - iZ)}{X - iZ}. \end{aligned} \quad (4.23)$$

As an example, we quote the first terms in the series expansion of \mathcal{A}_4^{ij} and \mathcal{A}_5

$$\mathcal{A}_4^{ij} = D^{ij} + b^{ij} [2i(c \cdot A \partial \bar{\phi}) + \bar{\phi}(c \cdot F)] + \dots, \quad \mathcal{A}_5 = -2D^{+-} - 4ib^{+-}(c \cdot A \partial \bar{\phi}) + \dots \quad (4.24)$$

Substituting the expressions (4.16), (4.19), (4.20) and the expression for $\mathcal{A}_{6\alpha\beta}$ with the functions g_i given by (4.21) into (4.14) (with taking into account (4.17)) and doing the harmonic integral we obtain

$$\begin{aligned} S_{bos} = \int d^4x & \left\{ -\frac{1}{2} \left[\phi + \frac{1}{2} \bar{\phi}^{-1} \left(1 - \frac{\tanh X}{X} \right) A^2 + (b^2 c^2)^{3/2} \tanh X \partial_{\alpha\dot{\alpha}} \bar{\phi} \partial^{\alpha\dot{\alpha}} \bar{\phi} \right] \square \bar{\phi} \right. \\ & + \frac{1}{4} \frac{D^2}{\cosh^2 X} - \frac{1}{16} F^2 \frac{\sinh^2 X}{X^2} + \frac{1}{2} \bar{\phi} (b \cdot D) (c \cdot F) \frac{\tanh^2 X}{X^2} + \frac{1}{4} b^2 (c \cdot F)^2 \bar{\phi}^2 \frac{\sinh^4 X}{X^4 \cosh^2 X} \\ & - i \left[(b \cdot D) (c \cdot A \partial \bar{\phi}) \frac{\tanh X}{X} + b^2 (c \cdot A \partial \bar{\phi}) (c \cdot F) \bar{\phi} \frac{\sinh^3 X}{X^3 \cosh X} \right] \left(\frac{\tanh X}{X} - \frac{2}{\cosh^2 X} \right) \\ & - (c \cdot A \partial \bar{\phi})^2 b^2 \frac{\sinh^2 X}{X^2} \left(\frac{\tanh X}{X} - \frac{2}{\cosh^2 X} \right)^2 + \frac{(A \partial \bar{\phi})^2 b^2 c^2}{\cosh^2 X} \left(\frac{\cosh X \sinh X - X}{X^2} \right)^2 \\ & \left. + i \sqrt{b^2 c^2} \frac{(F \cdot A \partial \bar{\phi}) \tanh X}{2} \frac{(\cosh X \sinh X - X)}{X} \right\}. \end{aligned} \quad (4.25)$$

Through the minimal SW map (see subsection 3.3) this action is simplified to

$$\begin{aligned} S_{bos} = \int d^4x & \left[-\frac{1}{2} \tilde{\phi} \square \bar{\phi} - \frac{1}{2} (b^2 c^2)^{3/2} \tanh X \partial_{\alpha\dot{\alpha}} \bar{\phi} \partial^{\alpha\dot{\alpha}} \bar{\phi} \square \bar{\phi} + \frac{1}{4} \frac{\tilde{D}^2}{\cosh^2 X} \right. \\ & \left. - \frac{1}{16} \tilde{F}^2 \cosh^2 X + \frac{1}{4} b^2 (c \cdot \tilde{F})^2 \bar{\phi}^2 \frac{\sinh^2 X}{X^2} + \frac{1}{2} \bar{\phi} (b \cdot \tilde{D}) (c \cdot \tilde{F}) \frac{\tanh X}{X} \right]. \end{aligned} \quad (4.26)$$

This action is invariant under the standard abelian gauge transformations. Turning off the deformation parameters we are left with the usual bosonic sector of the undeformed action. Performing the further field redefinition

$$d^{ij} = \frac{1}{\cosh^2 X} \tilde{D}^{ij} + \bar{\phi} (c \cdot \tilde{F}) b^{ij} \frac{\tanh X}{X}, \quad (4.27a)$$

$$\varphi = \frac{1}{\cosh^2 X} \left[\tilde{\phi} + (b^2 c^2)^{3/2} (\partial \bar{\phi})^2 \tanh X \right], \quad (4.27b)$$

the bosonic action can be transformed to the most simple form

$$S_{bos} = \int d^4x \cosh^2 X \left[-\frac{1}{2} \phi \square \bar{\phi} + \frac{1}{4} d^{ij} d_{ij} - \frac{1}{16} \tilde{F}^{\alpha\beta} \tilde{F}_{\alpha\beta} \right]. \quad (4.28)$$

From this expression it is obvious that we cannot disentangle the interaction between the gauge field and $\bar{\phi}$ by any field redefinition. This is similar to the singlet case [9, 7], where the scalar factor $(1 + 4I\bar{\phi})^2$ appears instead of $\cosh^2 X$. Note that the bosonic action (4.28) involves only squares c^2 and b^2 , so it preserves “Lorentz” $Spin(4) = SU(2)_L \times SU(2)_R$ symmetry and $SU(2)$ R-symmetry as in the singlet case, despite the fact that the deformation matrix (2.8) breaks both these symmetries down to $U(1)_L \times SU(2)_R$ and $U(1)$. This property is similar to what happens in the deformed Euclidean $\mathcal{N} = (1/2, 1/2)$ Wess-Zumino model where the deformation parameter $C^{\alpha\beta}$ also appears through its Lorentz-invariant square [3]. Note, however, that the fermionic completion of (4.28) explicitly includes both $c^{\alpha\beta}$ and b^{ik} [25], so these two symmetries are broken in the total action. This feature matches with the fact that Lorentz symmetry is broken in the action of deformed $\mathcal{N} = (1/2, 1/2)$ gauge theory, and also due to some fermionic terms [3].

4.2 Limiting cases

Let us expand the action (4.25) up to the first order in the deformation parameters b_{ij} and $c^{\alpha\beta}$ to compare it with the results of [7, 11, 12]. In this approximation the action reads

$$S_{bos} = \int d^4x_L \left[-\frac{1}{2} \phi \square \bar{\phi} + \frac{1}{4} D^2 + i c^{\alpha\beta} A_{\alpha\dot{\alpha}} \partial_{\dot{\beta}}^{\dot{\alpha}} \bar{\phi} D^{ij} b_{ij} - \frac{1}{16} F^2 + \frac{1}{2} \bar{\phi} D^{ij} b_{ij} c^{\alpha\beta} F_{\alpha\beta} \right]. \quad (4.29)$$

It can be checked that (4.29) coincides with the first-order action of references [7, 12] upon substituting there the product ansatz (2.8). The gauge transformations laws (3.33) in this case reduce to

$$\delta A_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} a, \quad (4.30a)$$

$$\delta \phi = 0, \quad (4.30b)$$

$$\delta \Psi_{\alpha}^i = -\frac{4}{3} b^{ij} c_{\alpha\beta} \bar{\Psi}_{j\dot{\alpha}} \partial^{\beta\dot{\alpha}} a, \quad (4.30c)$$

$$\delta D_{ij} = 2i b_{ij} c^{\alpha\beta} \partial_{\alpha\dot{\alpha}} \bar{\phi} \partial_{\dot{\beta}}^{\dot{\alpha}} a. \quad (4.30d)$$

These laws also precisely match with those given in [7, 12].

It would be also interesting to study the particular case of our results corresponding to the choice

$$c^{\alpha\beta} \neq 0 \quad b_{11} \neq 0, \quad b_{12} = b_{22} = 0 \rightarrow b^2 = 0. \quad (4.31)$$

This degenerate choice preserves $\mathcal{N} = (1, 1/2)$ supersymmetry [5, 12]. First we focus on the non-trivial gauge transformations (3.33), which for this choice are reduced to

$$\begin{aligned} \delta A_{\alpha\dot{\alpha}} &= \partial_{\alpha\dot{\alpha}} a, & \delta\phi &= 0, & \delta\Psi_{\alpha}^1 &= 0, \\ \delta\Psi_{\alpha}^2 &= \frac{4}{3} b_{11} c_{\alpha\beta} \bar{\Psi}_{\dot{\alpha}}^1 \partial^{\beta\dot{\alpha}} a, & \delta D_{11} &= 2i b_{11} c^{\alpha\beta} \partial_{(\alpha\dot{\alpha}} \bar{\phi} \partial_{\beta)}^{\dot{\alpha}} a, & \delta D_{12} &= \delta D_{22} = 0. \end{aligned} \quad (4.32)$$

Then we look at the action (4.29),

$$S_{bos} = \int d^4 x_L \left[-\frac{1}{2} \phi \square \bar{\phi} - \frac{1}{16} F^{\alpha\beta} F_{\alpha\beta} + \frac{1}{4} D^2 + i D^{11} b_{11} c^{\alpha\beta} \partial_{(\alpha\dot{\alpha}} \bar{\phi} A_{\beta)}^{\dot{\alpha}} + \frac{1}{2} \bar{\phi} D^{11} b_{11} c^{\alpha\beta} F_{\alpha\beta} \right]. \quad (4.33)$$

Since in our case $b^2 = 0$, this action is actually the *exact* form of (4.25) for the considered choice. To compare our results with those on [12] we have to use the minimal SW map for the choice (4.31), which is easy to calculate. In fact, the only non-trivial transformation is

$$D_{11} = \tilde{D}_{11} - 2i b_{11} c^{\alpha\beta} \tilde{A}_{(\alpha\dot{\alpha}} \partial_{\beta)}^{\dot{\alpha}} \bar{\phi}. \quad (4.34)$$

Performing it gives rise to the following action

$$S_{bos} = \int d^4 x_L \left[-\frac{1}{2} \tilde{\phi} \square \bar{\phi} - \frac{1}{16} \tilde{F}^{\alpha\beta} \tilde{F}_{\alpha\beta} + \frac{1}{4} \tilde{D}^2 + \frac{1}{2} \bar{\phi} \tilde{D}^{11} b_{11} c^{\alpha\beta} \tilde{F}_{\alpha\beta} \right]. \quad (4.35)$$

This action, as well as the gauge transformations (4.32), coincide with the expressions for the gauge transformations and bosonic sector of the action given in [12], up to a constant rescaling of the fields. Thus, proceeding from the $\mathcal{N} = (1, 1)$ superfield formalism, we have reproduced the results of [12] obtained within the $\mathcal{N} = 1$ superfield formalism. It is interesting to note that the redefinitions (4.27) are reduced to

$$\begin{aligned} \varphi &= \tilde{\phi}, \\ d^{11} &= D^{11}, \\ d^{12} &= D^{12}, \\ d^{22} &= D^{22} + \bar{\phi} c^{\alpha\beta} \tilde{F}_{\alpha\beta} b^{22}, \end{aligned} \quad (4.36)$$

and the action (4.35) becomes undeformed in terms of the new fields

$$S = \int d^4 x \left[-\frac{1}{2} \varphi \square \bar{\phi} + \frac{1}{4} d^{ij} d_{ij} - \frac{1}{16} \tilde{F}^{\alpha\beta} \tilde{F}_{\alpha\beta} \right], \quad (4.37)$$

which is obviously consistent with our result (4.28).

5 Unbroken $\mathcal{N} = (1, 0)$ supersymmetry

As we pointed out in the beginning of Sect. 4, there is no actual need to explicitly know the unbroken $\mathcal{N} = (1, 0)$ supersymmetry transformations, since the full action (with all fermionic terms taken into account [25]) is supersymmetric by construction. Nevertheless, for completeness, here we explain how to derive the $\mathcal{N} = (1, 0)$ transformations in the WZ gauge in the present case and give some examples of these transformations.

Unbroken supersymmetry is realized on V_{WZ}^{++} as

$$\delta V_{\text{WZ}}^{++} = (\epsilon^{+\alpha} \partial_{+\alpha} + \epsilon^{-\alpha} \partial_{-\alpha}) V_{\text{WZ}}^{++} - D^{++} \Lambda_c - [V_{\text{WZ}}^{++}, \Lambda_c]_{\star}, \quad (5.1)$$

where the star bracket, like in the previous consideration, is defined via the non-singlet Poisson structure with the deformation matrix (2.8) and Λ_c is the compensating gauge parameter which is necessary for preserving WZ gauge.

As in the case of deformed gauge transformations, for ensuring the correct undeformed limit, the parameter Λ_c should start with the parameter \mathbb{L} corresponding to the undeformed $\mathcal{N} = (1, 0)$ supersymmetry. Being expressed in chiral coordinates, \mathbb{L} is [9]

$$\mathbb{L} = l + (\bar{\theta}^+ l^-) + (\bar{\theta}^+)^2 l^{--}, \quad (5.2)$$

where

$$\begin{aligned} \lambda_{\epsilon} &= 2(\epsilon^{-} \theta^{+}) \bar{\phi}, \\ l^{-\dot{\alpha}} &= 4i(\epsilon^{-} \theta^{+}) \theta_{\alpha}^{-} \partial^{\alpha \dot{\alpha}} \bar{\phi} - 2\epsilon_{\alpha}^{-} A^{\alpha \dot{\alpha}} + 4(\epsilon^{-} \theta^{+}) \bar{\Psi}^{-\dot{\alpha}}, \\ l^{--} &= 2(\epsilon^{-} \Psi^{-}) + 2i\epsilon^{-\alpha} \theta^{-\beta} \partial_{\beta}^{\dot{\alpha}} A_{\alpha \dot{\alpha}} - 2(\epsilon^{-} \theta^{+}) (\theta^{-})^2 \square \bar{\phi} \\ &\quad + 4i(\epsilon^{-} \theta^{+}) \theta^{-\alpha} \partial_{\alpha \dot{\alpha}} \bar{\Psi}^{-\dot{\alpha}} + 2(\epsilon^{-} \theta^{+}) D^{--}. \end{aligned} \quad (5.3)$$

Let us for a while drop the undeformed part of the $\mathcal{N} = (1, 0)$ variation of V_{WZ}^{++} , which we denote by $\delta_0 V_{\text{WZ}}^{++}$ and which corresponds to substituting \mathbb{L} for Λ_c in (5.1) and discarding the last commutator term there. We denote by $\check{\delta} V_{\text{WZ}}^{++}$ the lowest-order non-singlet part of the transformations coming from the star commutator in (5.1) with \mathbb{L} as the gauge parameter. In

terms of superfield components, this part has the form

$$\begin{aligned}
\check{\delta} A_{\alpha\dot{\alpha}} &= 8\bar{\Psi}_{\dot{\alpha}}^{-}\epsilon^{-\beta}\bar{\phi}c_{\alpha\beta}b^{++}, \\
\check{\delta}\phi &= 8\left[2\Psi_{\alpha}^{-}\epsilon_{\beta}^{-}\bar{\phi}-A_{\alpha\dot{\alpha}}\bar{\Psi}^{-\dot{\alpha}}\epsilon_{\beta}^{-}\right]c^{\alpha\beta}b^{++}, \\
\check{\delta}\Psi^{-\alpha} &= 8\epsilon_{\beta}^{-}\left[(\bar{\Psi}^{-})^2-\bar{\phi}D^{--}\right]c^{\alpha\beta}b^{++}+2i\bar{\phi}\epsilon_{\gamma}^{-}\partial_{\beta\dot{\alpha}}A^{\gamma\dot{\alpha}}c^{\alpha\beta}b^{+-} \\
&\quad +2i\left[\bar{\phi}\epsilon_{\gamma}^{-}\partial_{\beta\dot{\alpha}}A^{\alpha\dot{\alpha}}+\partial_{\beta\dot{\alpha}}\bar{\phi}\epsilon^{-\alpha}A_{\gamma}^{\dot{\alpha}}\right]c^{\gamma\beta}b^{+-}, \\
\check{\delta}\bar{\Psi}_{\dot{\alpha}}^{-} &= 4i\bar{\phi}\partial_{\alpha\dot{\alpha}}\bar{\phi}\epsilon_{\beta}^{-}c^{\alpha\beta}b^{+-}, \\
\check{\delta}D^{--} &= -8i\left[\partial_{\alpha\dot{\alpha}}\bar{\phi}\epsilon_{\beta}^{-}\bar{\Psi}^{-\dot{\alpha}}+\bar{\phi}\epsilon_{\alpha}^{-}\partial_{\beta\dot{\alpha}}\bar{\Psi}^{-\dot{\alpha}}\right]c^{\alpha\beta}b^{+-},
\end{aligned} \tag{5.4}$$

where $\epsilon_{\alpha}^{-} = \epsilon_{\alpha}^i u_i^{-}$ and ϵ_{α}^i is the Grassmann $\mathcal{N} = (1, 0)$ transformation parameter. We observe here the same phenomenon as in the case of deformed gauge transformations in Sect. 3: these variations violate the WZ gauge due to the appearance of harmonic variables in the r.h.s., so one is led to properly modify \mathbb{L} . Moreover, in $\check{\delta}V_{WZ}^{++}$ there appear additional terms linear in the Grassmann variables

$$\check{\delta}V_{WZ}^{++} = 8\theta_{\alpha}^{+}\epsilon_{\beta}^{-}(\bar{\phi})^2c^{\alpha\beta}b^{++} - 8\bar{\theta}_{\dot{\rho}}^{+}A_{\alpha}^{\dot{\rho}}\epsilon_{\beta}^{-}\bar{\phi}c^{\alpha\beta}b^{++} + \dots \tag{5.5}$$

Thus, in contrast to the case of gauge transformations, the WZ gauge form of V_{WZ}^{++} proves to be broken by $\mathcal{N} = (1, 0)$ supersymmetry transformations with $\Lambda_c = \mathbb{L}$ not only in the harmonic sector but also in the Grassmann sector.

To solve this problem, we should promote the gauge parameter \mathbb{L} to the one providing correct transformations laws for the components fields in the WZ gauge. Thus we define

$$\Lambda_c = \mathbb{L} + F_{\epsilon} \tag{5.6}$$

and rewrite (5.1) in the following way

$$\begin{aligned}
\delta V_{WZ}^{++} &= (\epsilon^{+\alpha}\partial_{+\alpha} + \epsilon^{-\alpha}\partial_{-\alpha})V_{WZ}^{++} - D^{++}(\mathbb{L} + F_{\epsilon}) - [V_{WZ}^{++}, (\mathbb{L} + F_{\epsilon})]_{\star} \\
&= \delta_0 V_{WZ}^{++} + \check{\delta}V_{WZ}^{++} + \hat{\delta}V_{WZ}^{++},
\end{aligned} \tag{5.7}$$

where

$$\hat{\delta}V_{WZ}^{++} = -D^{++}(F_{\epsilon}) - [V_{WZ}^{++}, F_{\epsilon}]_{\star},$$

and

$$F_{\epsilon} = F + \bar{\theta}^{+}\bar{F}^{-} + (\bar{\theta}^{+})^2F^{--}$$

is the additional compensating gauge parameter intended for restoring the WZ gauge. The minimal set of terms needed to eliminate the improper harmonic and Grassmann dependence appearing in (5.7) and (5.4) amounts to the following form of F_ϵ :

$$\begin{aligned}
F &= \theta^{+\alpha} f_\alpha^-, \\
\bar{F}^{-\dot{\alpha}} &= \bar{g}^{-\dot{\alpha}} + 2i\theta_\alpha^- \theta^{+\beta} \partial^{\alpha\dot{\alpha}} f_\beta^- + \theta^{+\alpha} b_\alpha^{-\dot{\alpha}} + (\theta^+)^2 \bar{g}^{(-3)\dot{\alpha}}, \\
F^{--} &= g^{--} - (\theta^-)^2 \theta^{+\alpha} \square f_\alpha^- + i\theta^{-\alpha} \partial_{\alpha\dot{\alpha}} \bar{g}^{-\dot{\alpha}} + i\theta^{+\alpha} \theta^{-\beta} \partial_\beta^{\dot{\alpha}} b_{\alpha\dot{\alpha}}^- + \theta^{+\alpha} f_\alpha^{(-3)} \\
&\quad + i(\theta^+)^2 \theta^{-\alpha} \partial_{\alpha\dot{\alpha}} \bar{g}^{(-3)\dot{\alpha}} + (\theta^+)^2 X^{(-4)}.
\end{aligned} \tag{5.8}$$

The requirement that the terms in $\check{\delta}V_{\text{WZ}}^{++} + \hat{\delta}V_{\text{WZ}}^{++}$ which are linear in θ and $\bar{\theta}$ must vanish in order to restore the WZ gauge in the Grassmann sector gives rise to the following equations

$$\partial^{++} f^{-\alpha} + 4\bar{\phi} f_\beta^- c^{\alpha\beta} b^{++} + 8\epsilon_\beta^- (\bar{\phi})^2 c^{\alpha\beta} b^{++} = 0, \tag{5.9a}$$

$$\partial^{++} \bar{g}_{\dot{\alpha}}^- + 4A_{\alpha\dot{\alpha}} f_\beta^- c^{\alpha\beta} b^{++} + 8\epsilon_\beta^- A_{\alpha\dot{\alpha}} \bar{\phi} c^{\alpha\beta} b^{++} = 0. \tag{5.9b}$$

It is straightforward to check that the proper solution to these equations is given by

$$\begin{aligned}
f^{-\alpha} &= 2\bar{\phi} \epsilon_\beta^- \left[\varepsilon^{\alpha\beta} \left(\frac{\sinh X}{X} \cos Z - 1 \right) - c^{\alpha\beta} \sqrt{\frac{2}{c^2}} \frac{\sinh X}{X} \sin Z \right] \\
&\quad - 4\bar{\phi}^2 b^{--} \epsilon_\beta^+ \left\{ \varepsilon^{\alpha\beta} \sqrt{\frac{c^2}{2}} \frac{i}{X} \left[\frac{\sinh(X+iZ)}{X+iZ} - \frac{\sinh(X-iZ)}{X-iZ} \right] \right. \\
&\quad \left. - c^{\alpha\beta} \frac{1}{X} \left(\frac{1}{X-iZ} [\cosh(X-iZ) - 1] + \frac{1}{X+iZ} [\cosh(X+iZ) - 1] \right) \right\},
\end{aligned}$$

and

$$\bar{g}_{\dot{\alpha}}^- = A_{\alpha\dot{\alpha}} \bar{\phi}^{-1} f^{-\alpha}. \tag{5.10}$$

Note that these solutions are regular in $\bar{\phi}$, $c^{\alpha\beta}$ and b^{ik} as they should.

Using these expressions and requiring

$$\partial^{++} \delta \bar{\phi} = 0, \quad (\partial^{++})^2 \delta \bar{\Psi}_{\dot{\alpha}}^- = 0, \quad \partial^{++} \delta A_{\alpha\dot{\alpha}} = 0, \quad (\partial^{++})^2 \delta \Psi_\alpha^- = 0, \quad (\partial^{++})^3 \delta D^{--} = 0, \tag{5.11}$$

we can explicitly find other components of F_ϵ and restore the correct $\mathcal{N} = (1, 0)$ supersymmetry transformations preserving WZ gauge. We are planning to give the full set of these transforma-

tions in [25]. Here we quote only simplest ones

$$\delta\bar{\phi} = 0, \quad (5.12)$$

$$\delta\bar{\Psi}_{\dot{\alpha}}^i = \left[\frac{2i}{\sqrt{b^2 c^2}} \cosh X \sinh X c^{\alpha\beta} b^{ij} - i \cosh^2 X \varepsilon^{\alpha\beta} \varepsilon^{ij} \right] \epsilon_{j\beta} \partial_{\alpha\dot{\alpha}} \bar{\phi}, \quad (5.13)$$

$$\delta A_{\alpha\dot{\alpha}} = 8\bar{\phi} \epsilon^{i\beta} \bar{\Psi}_{\dot{\alpha}}^j b_{ij} c_{\alpha\beta} + 2\epsilon_{\alpha}^k \bar{\Psi}_{k\dot{\alpha}} X \coth X. \quad (5.14)$$

These variations form the algebra which is closed modulo a gauge transformation with the composite parameter $a_c = -2i(\epsilon \cdot \eta) \bar{\phi}$:

$$[\delta_{\epsilon}, \delta_{\eta}] \bar{\phi} = 0, \quad [\delta_{\epsilon}, \delta_{\eta}] \bar{\Psi}_{\dot{\alpha}}^j = 0,$$

$$[\delta_{\epsilon}, \delta_{\eta}] A_{\alpha\dot{\alpha}} = -2i(\epsilon \cdot \eta) (X \coth X) \partial_{\alpha\dot{\alpha}} \bar{\phi}.$$

The $\mathcal{N} = (1, 0)$ transformations are radically simplified for the degenerate choice (4.31) with $b^2 = 0$:

$$\delta\bar{\Psi}_{\dot{\alpha}}^- = -i \left(\varepsilon^{\alpha\beta} \varepsilon^{ij} - 4\bar{\phi} c^{\alpha\beta} b^{ij} \right) \epsilon_{j\beta} \partial_{\alpha\dot{\alpha}} \bar{\phi}, \quad \delta A_{\alpha\dot{\alpha}} = 2\epsilon_{\alpha}^k \bar{\Psi}_{k\dot{\alpha}} + 8\epsilon^{i\beta} \bar{\phi} \bar{\Psi}_{\dot{\alpha}}^j c_{\alpha\beta} b_{ij}. \quad (5.15)$$

6 Concluding remarks

In this paper we analyzed the model of a non-singlet Q-deformed $\mathcal{N} = (1, 1)$ supersymmetric U(1) gauge multiplet in harmonic superspace. We presented exact expressions for the gauge transformation of the fields, calculated the bosonic sector of the component action and gave a few examples of unbroken $\mathcal{N} = (1, 0)$ supersymmetry transformations in WZ gauge. All these results have been obtained for the special decomposition (1.6) of the $SU(2)_L \times SU(2)$ deformation matrix, namely $\hat{C}_{\alpha\beta}^{ij} = b^{ij} c_{\alpha\beta}$. This choice contains only six degrees of freedom, in contrast to the nine parameters of the generic non-singlet matrix (or two versus three after choosing an appropriate frame with respect to the broken $SU(2)_L \times SU(2)$ symmetry⁶). Let us summarize the key features of the ansatz (1.6).

- It provides a unique possibility to obtain all quantities in a closed compact form, in contrast to the generic non-singlet deformation case [7, 11, 12].
- It realizes the maximally symmetric non-singlet deformation, with the “Lorentz” U(1)_L and R-symmetry U(1) subgroups left unbroken. The generic non-singlet deformation fully breaks SU(2)_L and R symmetry.

⁶Using broken scale $O(1, 1)$ automorphism symmetry, one can further fix one parameter in both cases.

- With the choice $b^2 = 0$ it includes the important degenerate case which preserves 3/2 of the original $\mathcal{N} = (1, 1)$ supersymmetry (assuming a pseudoconjugation for the deformation parameters) [5, 12].
- It directly yields the well-known Seiberg ansatz after performing the reduction to $\mathcal{N} = (1/2, 1/2)$ superspace, with $c_{\alpha\beta}$ becoming the reduced deformation matrix ⁷.

The bosonic action has a structure similar to the singlet Q-deformed one calculated in [8, 9], in the sense that the gauge field develops a non-trivial interaction with one of the $\mathcal{N} = (1, 1)$ vector multiplet scalar field, i. e. $\bar{\phi}$. This interaction is the exact zero-fermion limit of the full component interaction lagrangian which, as follows from the superfield setup we started with, by construction breaks some fraction of the original supersymmetry. In the general case of $c^2 \neq 0, b^2 \neq 0$ the full component action should break the original $\mathcal{N} = (1, 1)$ supersymmetry by half, i.e. down to $\mathcal{N} = (1, 0)$, whereas the degenerate choice $b^2 = 0$ preserves the fraction 3/2 of $\mathcal{N} = (1, 1)$ supersymmetry, as noticed in [5] and discussed in more detail in a recent paper [12]. While the characteristic object of the Q-deformed $\mathcal{N} = (1, 1)$ gauge theory with the singlet deformation matrix is the polynomial factor $(1 + 4I\bar{\phi})$ [8, 9], where I is the deformation parameter, in the considered case there naturally appear *hyperbolic* functions of the argument $X = 2\bar{\phi}\sqrt{b^2c^2}$. Though the intrinsic reason of this mysterious appearance of the hyperbolic functions in the case of QNS-deformation with the product deformation matrix is so far unclear, it hopefully could be understood after clarifying possible relation of this sort of nilpotent deformation to some non-trivial backgrounds in string theory. Anyway, it is the manifestly supersymmetric harmonic superfield approach which allowed us to reveal these non-trivial structures at the component level: it would hardly be possible to exhibit them using from the very beginning the component approach or any approach based on the ordinary superfields.

More details of this special QNS-deformation of $\mathcal{N} = (1, 1)$ gauge theory, in particular, the total component action with fermions, will be given in our forthcoming paper [25]. As for the possible further developments, it would be interesting to study implications of this deformation

⁷Let us choose e.g. θ_α^1 as the left Grassmann co-ordinate of some $\mathcal{N} = (1/2, 1/2)$ subspace of the $\mathcal{N} = (1, 1)$ superspace, i.e. $\theta_1^\alpha \equiv \theta^\alpha$, assume the pseudoconjugation for all involved quantities as in [5], and fix the relevant broken automorphism $U(1)$ and $O(1, 1)$ symmetries of the $\mathcal{N} = (1, 1)$ superalgebra in such a way that $b_{ik} \equiv (b_{11}, b_{22}, b_{12}) = (1, b_{22}, 0)$. Then the deformation operator (2.2) for the choice (2.8) and $I = 0$ is reduced to $P = -\overleftarrow{\partial}_\alpha c^{\alpha\beta} \overrightarrow{\partial}_\beta - b_{22} \overleftarrow{\partial}_\alpha^2 c^{\alpha\beta} \overrightarrow{\partial}_\beta^2$, i.e. it is expressed as a sum of the mutually commuting chiral Poisson operators on two different $\mathcal{N} = (1/2, 1/2)$ subspaces of the $\mathcal{N} = (1, 1)$ superspace, with b_{22} being the “ratio” of two Seiberg deformation matrices. When $b_{22} = 0$, we face the case $b^2 = 0$ of Sect. 4.2, with only one $\mathcal{N} = (0, 1/2)$ supersymmetry broken. For $b_{22} \neq 0$, both $\mathcal{N} = (0, 1/2)$ supersymmetries are broken. The parameter b_{22} measures the breakdown of the second $\mathcal{N} = (0, 1/2)$ supersymmetry which is implicit in the $\mathcal{N} = (1/2, 1/2)$ superfield formulation based on the superspace $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. Recall that within the standard complex conjugation the reduction to $\mathcal{N} = (1/2, 1/2)$ superspace makes no sense since the latter is not closed under such conjugation [5].

in non-abelian $\mathcal{N} = (1, 1)$ gauge theory and in the models including hypermultiplets, along the lines of refs. [9, 10]. The study of quantum and geometric properties of these models, equally as revealing their possible phenomenological applications, e.g. as providing specific mechanism of the soft supersymmetry breaking, surely deserve further attention.

Finally, it would be interesting to treat the case of the generic non-singlet deformation matrix $\hat{C}_{\alpha\beta}^{ij}$ as a perturbation around the non-trivial ansatz (1.6) rather than around the undeformed limit. In this way one can hope to find a closed formulation of the general QNS-deformed theory.

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A Notation and conventions

In this Appendix we review the basics of $\mathcal{N} = (1, 1)$ Euclidean harmonic superspace which we use throughout the paper. For a deeper treatment of this subject we refer to [23, 24] (see also [5, 9, 10]).

The Euclidean harmonic superspace is defined as the product

$$\mathbb{H}\mathbb{R}^{4+2|8} = \mathbb{R}^{4|8} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)}, \quad (\text{A.1})$$

where $\mathbb{R}^{4|8}$ is the $\mathcal{N} = (1, 1)$ Euclidean superspace and $\mathrm{SU}(2)$ is the R-symmetry (automorphism) group of $\mathcal{N} = (1, 1)$ superalgebra. The topology of this superspace is $\mathbb{R}^{4|8} \times S^2$ and it is parametrized by the standard $(4 + 8)$ coordinates $(x^{\alpha\dot{\alpha}}, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i})$ of the superspace $\mathbb{R}^{4|8}$ and the $\mathrm{SU}(2)$ harmonic variables representing sphere $S^2 \sim \mathrm{SU}(2)/\mathrm{U}(1)$ and denoted by u_i^\pm .⁸ The harmonic variables are defined by the completeness relation

$$u_i^+ u_j^- - u_j^+ u_i^- = \varepsilon_{ij}. \quad (\text{A.2})$$

⁸Throughout the paper, Greek indices $\alpha, \dot{\alpha}$ are spinorial indices of the group $\mathrm{Spin}(4) = \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ and Latin indices i, j are the doublet indices of the R-symmetry group $\mathrm{SU}(2)$. Both sorts of indices are raised and lowered with the skew-symmetric metric $\varepsilon^{\alpha\beta}, \varepsilon^{\dot{\alpha}\dot{\beta}}, \varepsilon^{ik}$, e.g. $u_k^\pm = \varepsilon_{kj} u^\pm{}^j$, $u^\pm{}^k = \varepsilon^{kj} u_j^\pm$, where $\varepsilon_{12} = 1$, $\varepsilon^{\alpha\beta} \varepsilon_{\beta\gamma} = \delta_\gamma^\alpha$, etc.

Their symmetrized products

$$u_{(i_1}^+ \dots u_{i_n}^+ u_{j_1}^- \dots u_{j_m}^-)$$

form a complete basis of functions on the sphere S^2 .

While in the *central basis* the harmonic superspace is represented by the coordinates $(x^{\alpha\dot{\alpha}}, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}, u_i^\pm)$, its *analytic basis* is defined as the coordinate set $(x_A^{\alpha\dot{\alpha}}, \theta^{\alpha\pm}, \bar{\theta}^{\dot{\alpha}\pm}, u_i^\pm)$, where

$$\begin{aligned} x_A^{\alpha\dot{\alpha}} &= x^{\alpha\dot{\alpha}} - 4i\theta^{\alpha i} \bar{\theta}^{\dot{\alpha} j} u_{(i}^- u_{j)}^+, \\ \theta^{\alpha\pm} &= \theta^{\alpha k} u_k^\pm, \quad \bar{\theta}^{\dot{\alpha}\pm} = \bar{\theta}^{\dot{\alpha} k} u_k^\pm. \end{aligned} \quad (\text{A.3})$$

The covariant spinor derivatives in the analytic basis are defined as

$$\begin{aligned} D_\alpha^k u_k^+ &= D_\alpha^+ = \partial_{-\alpha}, & D_\alpha^k u_k^- &= D_\alpha^- = -\partial_{+\alpha} + 2i\bar{\theta}^{-\dot{\alpha}} \partial_{\alpha\dot{\alpha}}, \\ \bar{D}_{\dot{\alpha}}^k u_k^+ &= \bar{D}_{\dot{\alpha}}^+ = \bar{\partial}_{-\dot{\alpha}}, & \bar{D}_{\dot{\alpha}}^k u_k^- &= -\bar{D}_{\dot{\alpha}}^- = -\bar{\partial}_{+\dot{\alpha}} - 2i\theta^{-\alpha} \partial_{\alpha\dot{\alpha}}, \end{aligned} \quad (\text{A.4})$$

where

$$\partial_{\pm\alpha} = \frac{\partial}{\partial \theta^{\pm\alpha}}, \quad \bar{\partial}_{\pm\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}^{\pm\dot{\alpha}}}, \quad \partial_{\alpha\dot{\alpha}} = 2 \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$$

and $D_\alpha^k, \bar{D}_{\dot{\alpha}}^k$ are spinor derivatives in the central basis (their explicit form is not needed for us).

An important ingredient of the harmonic superspace is the covariant derivatives with respect to the harmonic variables. In the analytic basis they are

$$\begin{aligned} D_A^0 &= \partial^0 + \theta^{+\alpha} \partial_{+\alpha} - \theta^{-\alpha} \partial_{-\alpha} + \bar{\theta}^{+\dot{\alpha}} \bar{\partial}_{+\dot{\alpha}} - \bar{\theta}^{-\dot{\alpha}} \bar{\partial}_{-\dot{\alpha}}, \\ D_A^{++} &= \partial^{++} - 2i\theta^{+\alpha} \bar{\theta}^{+\dot{\alpha}} \partial_{\alpha\dot{\alpha}} + \theta^{+\alpha} \partial_{-\alpha} + \bar{\theta}^{+\dot{\alpha}} \bar{\partial}_{-\dot{\alpha}}, \\ D_A^{--} &= \partial^{--} - 2i\theta^{-\alpha} \bar{\theta}^{-\dot{\alpha}} \partial_{\alpha\dot{\alpha}} + \theta^{-\alpha} \partial_{+\alpha} + \bar{\theta}^{-\dot{\alpha}} \bar{\partial}_{+\dot{\alpha}} \end{aligned} \quad (\text{A.5})$$

with

$$\partial^0 = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}}, \quad \text{and} \quad \partial^{\pm\pm} = u^{\pm i} \frac{\partial}{\partial u^{\mp i}}.$$

They form an $SU(2)$ algebra:

$$[D^{++}, D^{--}] = D^0, \quad [D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm}.$$

The *left-chiral basis* of the harmonic superspace is represented by the coordinates $(x_L^{\alpha\dot{\alpha}}, \theta^{\pm\alpha}, \bar{\theta}^{\pm\dot{\alpha}}, u_i^\pm)$, where

$$x_L^{\alpha\dot{\alpha}} = x_A^{\alpha\dot{\alpha}} + 4i\theta^{-\alpha} \bar{\theta}^{+\dot{\alpha}}. \quad (\text{A.6})$$

In this basis, the differential operators used throughout the paper are written as

$$\begin{aligned} D_\alpha^+ &= \partial_{-\alpha} + 2i\bar{\theta}^{+\dot{\alpha}} \partial_{\alpha\dot{\alpha}}, & D_\alpha^- &= -\partial_{+\alpha} + 2i\bar{\theta}^{-\dot{\alpha}} \partial_{\alpha\dot{\alpha}}, & \bar{D}_{\dot{\alpha}}^+ &= \bar{\partial}_{-\dot{\alpha}}, & \bar{D}_{\dot{\alpha}}^- &= -\bar{\partial}_{+\dot{\alpha}}, \\ D^{++} &= \partial^{++} + \theta^{+\alpha} \partial_{-\alpha} + \bar{\theta}^{+\dot{\alpha}} \bar{\partial}_{-\dot{\alpha}}, & D^{--} &= \partial^{--} + \theta^{-\alpha} \partial_{+\alpha} + \bar{\theta}^{-\dot{\alpha}} \bar{\partial}_{+\dot{\alpha}}, \\ Q_\alpha^+ &= \partial_{-\alpha}, & Q_\alpha^- &= -\partial_{+\alpha}. \end{aligned} \quad (\text{A.7})$$

B Some useful harmonic integrals

In this Appendix we give explicit formulas for some harmonic integrals appearing in our calculations.

The integration over harmonics is fully specified by the two rules [24]

$$(a) \int du 1 = 1, \quad (b) \int du u_{(i_1}^+ \dots u_{i_n}^+ u_{j_1}^- \dots u_{j_m}^- = 0.$$

This means, in particular, that the harmonic integral of any object of the form $\partial^{++} f^{--}$ or $\partial^{--} f^{++}$ equals to zero, i.e. one can integrate by parts.

Using this property and eqs. (3.17), it is easy to compute

$$\int du (b^{+-})^{2(n+1)} = \frac{(-1)^{n+1}}{2n+3} \left(\frac{b^2}{2} \right)^{n+1}, \quad \int du (b^{+-})^{2n+1} = 0, \quad (\text{B.1})$$

whence, recalling that $Z = 2\bar{\phi}\sqrt{2c^2} b^{+-}$ and $X = 2\bar{\phi}\sqrt{b^2 c^2}$,

$$\int du \cos Z = \frac{\sinh X}{X}, \quad (\text{B.2})$$

$$\int du Z \sin Z = \frac{\sinh X - X \cosh X}{X}, \quad (\text{B.3})$$

$$\int du Z^2 \cos Z = \frac{2X \cosh X - 2 \sinh X - X^2 \sinh X}{X}, \quad (\text{B.4})$$

$$\int du Z^3 \sin Z = \frac{X^3 \cosh X - 3 X^2 \sinh X + 6 X \cosh X - 6 \sinh X}{X}, \quad (\text{B.5})$$

$$\int du Z^4 \cos Z = -4(6 + X^2) \cosh X + \frac{(24 + 12X^2 + X^4) \sinh X}{X}, \text{ etc.} \quad (\text{B.6})$$

The easiest way to compute integrals (B.3) - (B.6) is to introduce a real parameter α into (B.2) as $Z \rightarrow \alpha Z$, $X \rightarrow \alpha X$ and to repeatedly differentiate both sides of (B.2) with respect to α .

Another type of harmonic integrals include some object $A^{+-} = A^{(ik)} u_i^+ u_k^-$:

$$\int du A^{+-} (b^{+-})^{2n+1} = \frac{(-1)^{n+1}}{2(2n+3)} (A \cdot b) \left(\frac{b^2}{2} \right)^n, \quad (\text{B.7})$$

$$\int du (A^{+-})^2 (b^{+-})^{2n} = (-1)^{n+1} \frac{[A^2 b^2 + 2n (A \cdot b)^2]}{4 (2n+1) (2n+3)} \left(\frac{b^2}{2} \right)^{n-1}, \quad (\text{B.8})$$

$$\int du A^{+-} (b^{+-})^{2n} = 0, \quad \int du (A^{+-})^2 (b^{+-})^{2n+1} = 0. \quad (\text{B.9})$$

The vanishing of integrals in (B.9) directly follows from the fact that they should be $SU(2)$ invariants and the observation that it is impossible to form $SU(2)$ invariants from the given sets of traceless tensors $A^{(ik)}$ and $b^{(ik)}$. Integrals (B.7) and (B.8) can be directly computed using the identities of the type (3.16) following from the completeness relation (A.2).

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